The Darboux Transform in the Self-dual Grassmannian

Callum Kemp
University of Bath

Abstract

We develop a geometric understanding of the Darboux transform of isothermic submanifolds in the self-dual Grassmannian $G_k(\mathbb{C}^{2k})$. We relate a mutual pair of Darboux transforms to the common envelopes of a congruence of cyclides.

1 Introduction

Isothermic surfaces were studied extensively around the turn of the 20th Century by, among others, Bianchi [Bi1, Bi2], Blaschke [Bl], and Darboux [D]. Study of these was revived by Cieśliński-Goldstein-Sym [CGS], who showed that isothermic surfaces formed an integrable theorem. Burstall et al. [BDPP] then generalised this integrable structure to define isothermic submanifolds in certain other spaces, known as symmetric R-spaces.

In this paper, we aim to find an analogue of a result of Blaschke [Bl] (see [H, Section 3.1] for a full account of this). Blaschke shows that a Darboux pair of isothermic surfaces in the conformal 3-sphere envelop a common congruence of spheres and induce the same conformal structure on their domain. This has already been extended to the conformal n-sphere by Ma [M] but we will show that a similar result holds for maximal non-degenerate isothermic submanifolds in the Grassmannian.

In section 2, we will define isothermic submanifolds of the Grassmannian and their Darboux transforms and collect several results about Darboux transforms and curved flats.

Finally, in section 3, we define our special submanifolds, called cyclides, as well as defining a generalised conformal structure. This allows us to state our main theorem linking Darboux pairs to envelopes of cyclide congruences via curved flats.

2 Isothermic submanifolds of the Grassmannian

2.1 The Grassmannian

Let $\mathfrak{g} = (n, \mathbb{C})$ and G be its adjoint group isomorphic to $PSL(n, \mathbb{C})$. We denote the Grassmannian of k-dimensional subspaces of $\mathbb{C}^n$ by $G_k(\mathbb{C}^n)$. The group G acts transitively on $G_k(\mathbb{C}^n)$ so that we may view it as a homogeneous space. We may use this fact to study the geometry of $G_k(\mathbb{C}^n)$ using the Lie theory of G. As an example of this, the solder form gives an isomorphism for each $V \in G_k(\mathbb{C}^n)$.

$$\begin{aligned}\beta_V : \operatorname{Hom}(V, \mathbb{C}^n/V) \cong \mathfrak{g}/\operatorname{stab}(V) &\to T_V G_k(\mathbb{C}^n);\\ X + \operatorname{stab}(V) &\mapsto \left.\frac{d}{dt}\right|_{t=0} \exp(tX)\cdot V,\end{aligned} \tag{1}$$

where $\operatorname{stab}(V) \leq \mathfrak{g}$ denotes the infinitesimal stabiliser of V. We can consider $\operatorname{stab}(V)^\perp$ the polar of $\operatorname{stab}(V)$ with respect to the Killing form of $\mathfrak{g}$. In fact, this is dual to $\mathfrak{g}/\operatorname{stab}(V)$ and so we make the identifications:

$$\operatorname{stab}(V)^\perp \cong \operatorname{Hom}(\mathbb{C}^n/V, V) \cong T_V^* G_k(\mathbb{C}^n) \tag{2}$$

Note that this is an abelian subalgebra of $\mathfrak{g}$ which will be important for the definition of isothermic submanifold.

Extending the concept of projective duality, $G_k(\mathbb{C}^n)$ has a natural dual space. Let $V \in G_k(\mathbb{C}^n)$. Then we say $W \leq \mathbb{C}^n$ is complementary to V if $V \oplus W = \mathbb{C}^n$. Clearly then $W \in G_{n-k}(\mathbb{C}^n)$. This choice allows us to identify $\mathbb{C}^n/V \cong W$ and similarly $\mathbb{C}^n/W \cong V$.

Definition 2.1 (Space of complementary pairs)**.** *Let*

$$Z := \{(V, W) \in G_k(\mathbb{C}^n) \times G_{n-k}(\mathbb{C}^n) | V \oplus W = \mathbb{C}^n\}. \tag{3}$$

We call this the ***space of complementary pairs****.*

Proposition 2.2. *Z is a pseudo-Riemannian symmetric G-space.*

Proof. Firstly, we note that G acts transitively on $G_k(\mathbb{C}^n)$. Moreover, $\exp(\operatorname{stab}(V)^\perp) \leq \operatorname{Stab}(V) \leq G$ acts transitively on the set of $W \in G_{n-k}(\mathbb{C}^n)$ complementary to V. Thus G acts transitively on Z.

Let $\mathfrak{h}_{V,W} := \operatorname{stab}(V) \cap \operatorname{stab}(W)$, $\mathfrak{m}_{V,W} := \operatorname{stab}(V)^\perp \oplus \operatorname{stab}(W)^\perp$. Then $\mathfrak{h}_{V,W}$ is the infinitesimal stabiliser of $(V, W) \in Z$ and $[\mathfrak{h}_{V,W}, \mathfrak{m}_{V,W}] \subset \mathfrak{m}_{V,W}, [\mathfrak{m}_{V,W}, \mathfrak{m}_{V,W}] \subset \mathfrak{h}_{V,W}$. Thus Z is a symmetric space. □

2.2 Isothermic submanifolds

Let $f : \Sigma \to G_k(\mathbb{C}^n)$ be some map from a manifold Σ. We can then shift our viewpoint and consider f as a bundle:

$$f_x := \operatorname{stab}(f(x)) \tag{4}$$

Equally, we can define a bundle $f^\perp$ by $f_x^\perp := (f_x)^\perp$. Both of these are subbundles of the trivial bundle $\underline{\mathfrak{g}} = \Sigma \times \mathfrak{g}$. In particular, we have a trivial connection d on $\underline{\mathfrak{g}}$ allowing us to make the following definition.

Definition 2.3. *Let $f : \Sigma \to G_k(\mathbb{C}^n)$. We call f* ***isothermic*** *if there exists a non-zero 1-form $\eta \in \Omega^1(f^\perp)$ such that $d\eta = 0$. Then (f, η) is an* ***isothermic submanifold*** *if f immerses.*

Using (2) we can view η as a $f^{-1}T^*G_k(\mathbb{C}^n)$ valued 1-form, meaning we can contract it with df to give 2-tensor $q_f(X, Y) := \eta_X(df_Y)$. In fact, q_f is symmetric [**?**]*Proposition 6.1 so we make the definition:

Definition 2.4. *We call q_f the* ***quadratic form associated to*** (f,η)*. We say* (f,η) *is* ***non-degenerate*** *if q_f is non-degenerate.*

Isothermic submanifolds are an example of an integrable system. In geometric terms, this manifests as a family of flat connections. Let $f:\Sigma\to G_k(\mathbb{C}^n)$ be any map and $\eta\in\Omega^1(f^\perp)$. Define $\nabla^t := d+t\eta$. Then the curvature of ∇^t is:

$$R^{\nabla^t} = R^d + td\eta + \frac{t^2}{2}[\eta\wedge\eta]. \tag{5}$$

Now d is flat and $f^\perp$ is a bundle of abelian subalgebras of $\mathfrak{g}$ so ∇^t is flat if, and only if, (f,η) is isothermic.

Using this family of connections allows us to define transformations of isothermic submanifolds. While it is possible to define the T-transform using this data, this paper will focus on the Darboux transform.

Definition 2.5 (Darboux transform)**.** *Let (f,η) be an isothermic submanifold of $G_k(\mathbb{C}^n)$ and $m\in\mathbb{R}\backslash\{0\}$. Then $\hat{f}:\Sigma\to G_{n-k}(\mathbb{C}^n)$ is a* ***Darboux transform*** *of f with parameter m if:*

- *Any section ζ of $\hat{f}$ has $\nabla^m\zeta\in\Omega^1_\Sigma(\hat{f})$,*
- *f and $\hat{f}$ are pointwise complementary.*

We will refer to $f,\hat{f}$ as a Darboux pair.

Proposition 2.6 ([BDPP, Theorem 3.10])**.** *If $\hat{f}$ is a Darboux transform of f then f is also a Darboux transform of $\hat{f}$.*

A key result that we will use is that a Darboux pair is precisely a curved flat into Z. Curved flats are another integrable system developed by Ferus-Pedit [FP] and can be defined into any symmetric space. We shall focus narrow our focus to curved flats in Z, however. For $(f,\hat{f}):\Sigma\to Z$ we can define $\mathcal{N}\in\Omega^1(f^\perp\oplus\hat{f}^\perp)$ by:

$$\mathcal{N} := \pi_{\hat{f}^\perp}\circ df + \pi_{f^\perp}\circ d\hat{f}. \tag{6}$$

Definition 2.7 (Curved flat)**.** *Let $(f,\hat{f}):\Sigma\to Z$. Then ϕ is a* ***curved flat*** *if $\mathrm{Im}\,\mathcal{N}_x$ is an abelian subalgebra of $f_x^\perp\oplus\hat{f}_x^\perp$ for all $x\in\Sigma$.*

Proposition 2.8 ([BDPP, Theorem 5.8])**.** *A map $(f,\hat{f}):\Sigma\to Z$ is a curved flat if, and only if, $f,\hat{f}$ are a Darboux pair of isothermic maps.*

If we restrict our attention to non-degenerate isothermic submanifolds of maximal dimension we can obtain a further result.

Definition 2.9 (Cartan subspace)**.** *Let $(V,W)\in Z$, and let $\mathfrak{c}\leq\mathrm{stab}(V)^\perp\oplus\mathrm{stab}(W)^\perp$ be a maximal abelian subalgebra all of whose elements are semisimple. Then we call $\mathfrak{c}$ a* ***Cartan subspace*** *of $\mathrm{stab}(V)^\perp\oplus\mathrm{stab}(W)^\perp$.*

Proposition 2.10 ([BDPP, Theorem 6.4 and Proposition 6.5])**.** *Two maps $f,\hat{f}$ are a Darboux pair of maximal non-degenerate isothermic submanifolds if, and only if, $\mathrm{Im}\,\mathcal{N}_x$ is a Cartan subspace for all $x\in\Sigma$.*

3 Cyclides

In this section, we will demonstrate a geometric interpretation for the Darboux transform of maximal non-degenerate isothermic submanifolds in the self-dual Grassmannian $G_k(\mathbb{C}^{2k})$. Such a Darboux pair will be seen to be the common envelopes of a congruence of submanifolds, the form of which we will define.

The motivation for our definition will be that a Cartan subspace for Z defines a splitting of two complementary planes $V, W \in G_k(\mathbb{C}^{2k})$ into paired lines.

Proposition 3.1. *Let $(V, W) \in Z \subset (G_k(\mathbb{C}^{2k}))^2$. Then any Cartan subspace of $\mathfrak{m}_{V,W}$ is of the form:*

$$\mathfrak{c} = \langle E_i + \hat{E}_i | i = 1, \ldots, k\rangle, \tag{7}$$

for $E_i \in L_i^* \otimes \hat{L}_i, \hat{E}_i \in \hat{L}_i^* \otimes L_i$, where $V = \bigoplus_{i=1}^k L_i, W = \bigoplus_{i=1}^k \hat{L}_i$.

Proof. From [LM, Theorem 4.1] we see that all Cartan subspaces of $\mathfrak{m}_{V,W}$ are conjugate. We therefore need only show that one such $\mathfrak{c}$ is a Cartan subspace. Firstly, we note $\mathfrak{c}$ is clearly abelian. If $X \in \mathfrak{m}_{V,W} = \mathrm{Hom}(V, W) \oplus \mathrm{Hom}(W, V)$ such that $[X, \mathfrak{c}] = 0$ then we see:

$$[X, E_i + \hat{E}_i] = 0, \tag{8}$$

for each i. Using our splitting $\mathbb{C}^{2n} = \bigoplus_{i=1}^k L_i \oplus \bigoplus_{i=1}^k \hat{L}_i$, we obtain a splitting of $\mathfrak{m}_{V,W} = \bigoplus_{1 \le i,j \le k} L_i^* \otimes \hat{L}_j \oplus \bigoplus_{1 \le i,j \le k} \hat{L}_i^* \otimes L_j$. Then, projecting onto these factors in (8) tells us that $X \in \mathfrak{c}$. Thus $\mathfrak{c}$ is a maximal abelian. Each $E_i + \hat{E}_i$ is a semisimple endomorphism and, as $\mathfrak{c}$ is abelian, any linear combination of these elements is also semisimple. □

We now wish to endow $G_k(\mathbb{C}^{2k})$ with a structure akin to the conformal structure of the n-sphere. We adapt the definition of Gindikin-Kaneyuki [GK] to $G_k(\mathbb{C}^{2k})$:

Definition 3.2 (Generalised conformal structure)**.** *The* ***generalised conformal structure*** *of $G_k(\mathbb{C}^{2k})$ at V is:*

$$\mathcal{C}_V := \{X \in T_V G_k(\mathbb{C}^{2k}) | \mathrm{rank}(X) < \dim V\}. \tag{9}$$

In other words, the generalised conformal structure is the set of all elements of $\mathrm{Hom}(V, \mathbb{C}^{2k}/V)$ that fail to be isomorphisms

Let $V_1 \oplus \cdots \oplus V_k = \mathbb{C}^{2k}$ be a splitting into 2-dimensional subspaces. Then:

$$\Phi : \mathbb{P}(V_1) \times \cdots \times \mathbb{P}(V_k) \hookrightarrow G_k(\mathbb{C}^{2k}); (L_1, \ldots, L_k) \mapsto L_1 \oplus \cdots \oplus L_k \tag{10}$$

is an embedded submanifold. In fact, the domain is compact so we only need to show that it is an injective immersion. It is naturally injective and by (1)

$$\mathrm{Im}(d\Phi)_{(L_1,\ldots,L_k)} = \bigoplus_{i=1}^k \mathrm{Hom}(L_i, V_i/L_i) \subset \mathrm{Hom}(\bigoplus_{i=1}^k L_i, \mathbb{C}^{2k}/\bigoplus_{i=1}^k L_i). \tag{11}$$

We note that these tangent spaces intersect with the generalised conformal structure as the union of k linearly independent hyperplanes:

$$H_i := \bigoplus_{j \ne i} \mathrm{Hom}(L_j, V_j/L_j). \tag{12}$$

If, for each i, we take some $\hat{L}_i \leq V_i$ distinct from L_i, then $\hat{L} := \bigoplus_{i=1}^k \hat{L}_i$ is complementary to $L := \bigoplus_{i=1}^k L_i$. Thus we can make some identifications of the tangent spaces as in [ref tangent space with complement] $\mathrm{Hom}(V_i/L_i, L_i) \cong \mathrm{Hom}(\hat{L}_i, L_i), \mathrm{Hom}(V_i/\hat{L}_i, \hat{L}_i) \cong \mathrm{Hom}(L_i, \hat{L}_i)$. Then the general conformal hyperplanes at L and $\hat{L}$ are naturally paired. Let

$$\hat{H}_i := \bigoplus_{j \neq i} \mathrm{Hom}(L_j, \hat{L}_j). \tag{13}$$

Theorem 3.3. *Let $f, \hat{f} : \Sigma \to G_k(\mathbb{C}^{2k})$ be pointwise complementary. Then $f, \hat{f}$ are a Darboux pair of maximal non-degenerate isothermic surfaces if, and only if, they envelop a common cyclide congruence, they induce the same generalised conformal structure on $T\Sigma$ and $d\hat{f} \circ df^{-1}(H_i) = \hat{H}_i$.*

Proof. Let $f, \hat{f} : \Sigma \to G_k(\mathbb{C}^{2k})$ be a Darboux pair of maximal non-degenerate isothermic submanifolds. Then $(f, \hat{f}) : \Sigma \to Z$ is a curved flat. Now Proposition 3.1 tells us that $\mathcal{N} = \sum_{i=1}^k \omega_i \otimes (E_i + E_{-i})$ for $E_i \in \hat{L}_i^* \otimes L_i, \hat{E}_i \in L_i^* \otimes \hat{L}_i$ and $f = \bigoplus_{i=1}^k L_i, \hat{f} = \bigoplus_{i=1}^k \hat{L}_i$. Thus,

$$df = \sum_{i=1}^k \omega_i \otimes E_i, \; d\hat{f} = \sum_{i=1}^k \omega_i \otimes \hat{E}_i. \tag{14}$$

The cyclide congruence C defined by $V_i := L_i \oplus \hat{L}_i$ then contains $f, \hat{f}$ at each point and has tangent spaces:

$$T_f C = \langle E_i | i = 1, \ldots, k \rangle, \; T_{\hat{f}} C = \langle \hat{E}_i | i = 1, \ldots, k \rangle. \tag{15}$$

Thus, $f, \hat{f}$ envelop C and induce the same generalised conformal structure on $T\Sigma$ given by $\bigcup_{i=1}^k \mathrm{Ker}\, \omega_i$. Moreover, $d\hat{f} \circ df^{-1}$ sends E_i to $\hat{E}_i$ and therefore it preserves the generalised conformal hyperplanes.

Conversely, let $f, \hat{f}$ envelop a common cyclide congruence C defined by V_i for $i = 1, \ldots, k$. Let $L_i := V_i \cap f, \hat{L}_i := V_i \cap \hat{f}$. Then

$$df = \sum_{i=1}^k \omega_i \otimes E_i, \; d\hat{f} = \sum_{i=1}^k \hat{\omega}_i \otimes \hat{E}_i. \tag{16}$$

with $E_i \in \hat{L}_i^* \otimes L_i, \hat{E}_i \in L_i^* \otimes \hat{L}_i$. These have generalised conformal hyperplanes $H_i := \bigoplus_{j \neq i} \mathrm{Hom}(\hat{L}_j, L_j)$, $\hat{H}_i := \bigoplus_{j \neq i} \mathrm{Hom}(L_j, \hat{L}_j)$. Then, the induced generalised conformal structures on $T\Sigma$ are given by $\bigcup_{i=1}^k \mathrm{Ker}\, \omega_i$ and $\bigcup_{i=1}^k \mathrm{Ker}\, \hat{\omega}_i$ respectively. If these are the same then each $\hat{\omega}_i$ must be pointwise the scale of some ω_j. If we require $d\hat{f} \circ df^{-1}(H_i) = \hat{H}_i$ this forces $\hat{\omega}_i = \lambda_i \omega_i$ for some $\lambda_i : \Sigma \to \mathbb{C}$. Thus, $\mathcal{N} := \pi_{\hat{f}^\perp} \circ df + \pi_{f^\perp} \circ d\hat{f} = \sum_{i=1}^k \omega_i \otimes E_i + \lambda_i$ and by rescaling $\hat{E}_i$ we see that $\mathrm{Im}\, \mathcal{N}_x$ is a Cartan subspace for all $x \in \Sigma$. Therefore $f, \hat{f}$ are a Darboux pair of maximal non-degenerate isothermic submanifolds. $\square$

References

[Bi1] Luigi Bianchi, Ricerche sulle superficie isoterme e sulla deformazione delle quadriche, Ann. Mat. **11** (1905), 93-157.

[Bi2] Luigi Bianchi, Complimenti alle ricerche sulle superficie isoterme, Ann. Mat. **12** (1905), 19-54.

[Bl] Wilhelm Blaschke, *Kreis und Kugel*, 2te Aufl., Walter de Gruyter & Co., Berlin 1956.

[BDPP] Francis Burstall, Neil Donaldson, Franz Pedit and Ulrich Pinkall, Isothermic submanifolds of symmetric R-spaces, J. Reine Angew. Math. **660** (2011), 191-243.

[CGS] Jan Cieśliński, Piotr Goldstein and Antoni Sym, Isothermic surfaces in $\mathbf{E}^3$ as soliton surfaces, Phys. Lett. A **205** (1995), no.1, 37-43.

[D] Gaston Darboux, Sur les surfaces isothermiques, Ann. Sci. École Norm. Sup. (3) **16** (1899), 491-508.

[FP] Dirk Ferus and Franz Pedit, Curved flats in symmetric spaces, Manuscr. Math. **91** (1996), no. 4, 445?454.

[GK] Simon Gindikin and Soji Kaneyuki, On the automorphism group of the generalized conformal structure of a symmetric R-space, Diff. Geom. Appl. **8** (1998), 21?33.

[H] Udo Hertrich-Jeromin, *Introduction to Möbius differential geometry*, London Math. Soc. Lect. Note Ser. **300**, Cambridge University Press, Cambridge 2003.

[LM] J. Lepowsky and G. W. McCollum, Cartan subspaces of symmetric Lie algebras, Trans. Amer. Math. Soc. **216** (1976), 217?228.

[M] Xiang Ma, Isothermic and S-Willmore surfaces as solutions to a problem of Blaschke, Results Math. **48** (2005), no. 3-4, 301-309.

University of Bath
Bath BA2 7AY
UK
E-mail address: c.kemp@bath.ac.uk